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Maximal symmetry groups of quantum relativistic equations

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Abstract. We have used Lie's extended group method to obtain the maximal symmetry groups of the Dirac equation for finite mass spin-$\frac{1}{2}$ particles and the Weyl equation for zero mass spin-$\frac{1}{2}$ particles. In both cases the maximal symmetry group is an infinite parameter Lie group having an invariant subgroup also of an infinite number of generators. The corresponding factor group for the Dirac equation is an eleven-parameter Lie group isomorphic to the Weyl group. In the case of the Weyl equation the corresponding factor group is a sixteen-parameter Lie group containing a proper subgroup isomorphic to the conformal group.

1. Introduction

The importance of the ten-parameter inhomogeneous Lorentz group (Wigner 1939) in relativistic quantum physics is that the Dirac equation for finite mass spin-$\frac{1}{2}$ particles, the Weyl equation for zero mass spin-$\frac{1}{2}$ particles and the Maxwell equation for zero mass spin-1 particles can be obtained (Elliott and Dawber 1979, Lyubarskii 1960) from different irreducible representations of the inhomogeneous Lorentz group (also known as the Poincaré group). Yet it has been known from the beginning of this century (Bateman 1910) that Maxwell's electrodynamic equations are invariant under the larger fifteen-parameter conformal group of Minkowski space. Gross (1964) showed the norm invariance of quantal relativistic equations for zero mass particles under the conformal group. Fulton *et al* (1962) discussed the role of conformal symmetry in different branches of physics including the Klein-Gordon equation for scalar fields and the Dirac equation.

In a previous publication (Rudra 1986) we obtained the maximal symmetry group of the Hamilton-Jacobi equation for a non-quantal particle, both of finite mass and zero mass. Here we are interested to know what the maximal symmetry groups of the multicomponent equations of Dirac and Weyl are. By maximal symmetry group we mean the maximal Lie group for the transformations of the spacetime coordinates and the multicomponent wavefunction that keeps the form of the differential equation invariant. We have used Lie's extended group method (Hamermesh 1984, Olver 1976, Rudra 1984, Sattinger 1977) to obtain these symmetry groups. Recently this method has been successfully used (Leach 1981, Prince and Leach 1980, Wulfman and Wybourne 1976) to obtain the maximal symmetry groups for the classical harmonic oscillator and Kepler motion. Our analysis of the two quantal relativistic equations shows that for each of them the maximal symmetry group is an infinite parameter Lie group having an infinite parameter invariant subgroup. The corresponding factor group for the Dirac equation is the eleven-parameter Weyl group consisting of the generators

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of the Poincaré group and the scale transformation of the four-component wavefunction. The factor group for the Weyl equation for zero mass spin-$\frac{1}{2}$ particles is a sixteen-parameter Lie group containing a proper subgroup isomorphic to the conformal group.

In § 2 we have described Lie's extended group method for obtaining the maximal symmetry group of a set of differential equations. Since our equations of interest are linear first-order partial differential equations, we have considered this case in some detail. In §§ 3 and 4 we have applied the method to obtain the generators of the maximal symmetry groups for the Dirac and Weyl equations.

2. Lie's extended group method

We now describe Lie's extended group method and develop it in a form suitable for linear first-order partial differential equations of quantum relativistic physics.

We consider a set of partial differential equations

$$\Delta^{\alpha}(q, \Psi; r) = 0 \qquad \alpha = 1, \ldots, p \tag{1}$$

in s dependent variables Ψ^k, $k = 1, \ldots, s$, and n independent variables q^i, $i = 1, \ldots, n$. Here r denotes the highest order of partial derivatives of Ψ^k. We first construct a space of all variables and derivatives q^i, Ψ^k and Ψ^k_J, where

$$\Psi^k_J = \partial^{|J|}\Psi^k \left(\prod_{i=1}^{n} (\partial q^i)^{j_i}\right)^{-1} \qquad \text{with } J \equiv (j_1, \ldots, j_n), |J| = \sum_i j_i \tag{2}$$

j_i being non-negative integers. If

$$X = \sum_i \xi^i(q, \Psi)\partial/\partial q^i + \sum_k \varphi_k(q, \Psi)\partial/\partial\Psi^k \tag{3}$$

is the generator in the product space (q, Ψ) then the rth extension $X^{(r)}$ of X is given by

$$X^{(r)} = X + \sum_k \sum_{1 \leq |J| \leq r} \varphi^J_K(q, \Psi, \Psi_J)\partial/\partial\Psi^k_J. \tag{4}$$

Here

$$\varphi^J_k = D^J\left(\varphi_k - \sum_i \Psi^k_i \xi^i\right) + \sum_i \Psi^k_{J,i}\xi^i \tag{5}$$

where

$$\Psi^k_i = \partial\Psi^k/\partial q^i \qquad (J, i) \equiv (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_n) \tag{6}$$

and

$$D^J = \prod_{i=1}^{n} D_i^{j_i} \qquad \text{with } D_i = \partial/\partial q^i + \sum_k \sum_{0 \leq |J| \leq r} \Psi^k_{J,i}\partial/\partial\Psi^k_J. \tag{7}$$

The system of partial differential equations (1) has the maximal symmetry group G with generators X if

$$X^{(r)}\Delta^{\alpha}(q, \Psi; r) = 0 \qquad \alpha = 1, \ldots, p. \tag{8}$$

It is to be mentioned that in equations (3)–(8) the q^i, Ψ^k and Ψ^k_J are to be considered as independent variables. On the left-hand side of equation (8) we use equation (1) and separately equate to zero the coefficients of different order partial derivatives of Ψ^k and their products and thus obtain a set of partial differential equations for ξ and φ. The solutions of these partial differential equations give us the most general form of X and hence the maximal symmetry group.

In the case of linear first-order partial differential equations of quantal relativistic physics

$$\Delta^k \equiv \Psi^k_\tau - \sum_{m\alpha} a^\alpha_{km} \Psi^m_\alpha - \sum_m a^0_{km} \Psi^m = 0 \qquad k = 1, \ldots, n \tag{9}$$

where the coordinates are τ and $x^\alpha (\alpha = 1, 2, 3)$. We take the general form of the generator as

$$X = \xi^\tau \partial/\partial\tau + \sum_\alpha \xi^\alpha \partial/\partial x^\alpha + \sum_k \varphi_k \partial/\partial\Psi^k. \tag{10}$$

In equation (9) the subscripts τ and α denote partial differentiation with respect to τ and x^α respectively. Equation (8) now becomes

$$\begin{aligned}
X^{(1)}\Delta \equiv \Bigg(& \varphi_{l;\tau} - \sum_m a^0_{lm}\varphi_m - \sum_{m\alpha} a^\alpha_{lm}\varphi_{m;\alpha} + \sum_{mk} \varphi_{l;\Psi^m} a^0_{mk}\Psi^k \\
& - \xi^\tau_\tau \sum_m a^0_{lm}\Psi^m + \sum_{mk\alpha} a^0_{mk} a^\alpha_{lm} \xi^\tau_\alpha \Psi^k - \sum_{mrk} a^0_{lk} a^0_{mr} \xi^\tau_{\Psi^m} \Psi^k \Psi^r \\
& - \sum_m \xi^\tau (\partial a^0_{lm}/\partial\tau)\Psi^m - \sum_{m\alpha} \xi^\alpha (\partial a^0_{lm}/\partial x^\alpha)\Psi^m \Bigg) \\
& + \sum_{m\alpha} \Psi^m_\alpha \Bigg[-\delta_{ml}\Bigg(\xi^\alpha_\tau + \sum_{rk} a^0_{rk} \xi^\alpha_{\Psi^r} \Psi^k \Bigg) - a^\alpha_{lm}\xi^\tau_\tau + \sum_\beta a^\beta_{lm} \xi^\alpha_\beta \\
& - \sum_k a^\alpha_{lk} \varphi_{k;\Psi^m} + \sum_k a^\alpha_{km} \varphi_{l;\Psi^k} + \sum_{k\beta} a^\alpha_{km} a^\beta_{lk} \xi^\tau_\beta \\
& + \sum_{rk} a^0_{rk} a^\alpha_{lr} \xi^\tau_{\Psi^m} \Psi^k - \sum_{rk} a^0_{lk} a^\alpha_{rm} \xi^\tau_{\Psi^r} \Psi^k - \sum_{rk} a^0_{kr} a^\alpha_{lm} \xi^\tau_{\Psi^k} \Psi^r \\
& - \xi^\tau (\partial a^\alpha_{lm}/\partial\tau) - \sum_\beta \xi^\beta (\partial a^\alpha_{lm}/\partial x^\beta) \Bigg] \\
& + \tfrac{1}{2} \sum_{mk\alpha\beta} \Psi^m_\alpha \Psi^k_\beta \Bigg(-\delta_{ml} \sum_r a^\beta_{rk} \xi^\alpha_{\Psi^r} + a^\beta_{lm} \xi^\alpha_{\Psi^k} + \sum_r a^\alpha_{rm} a^\beta_{lr} \xi^\tau_{\Psi^k} \\
& - \sum_r a^\alpha_{lm} a^\beta_{rk} \xi^\tau_{\Psi^r} - \delta_{kl} \sum_r a^\alpha_{rm} \xi^\beta_{\Psi^r} + a^\alpha_{lk} \xi^\beta_{\Psi^m} + \sum_r a^\beta_{rk} a^\alpha_{lr} \xi^\tau_{\Psi^m} \\
& - \sum_r a^\beta_{lk} a^\alpha_{rm} \xi^\tau_{\Psi^r} \Bigg) = 0.
\end{aligned} \tag{11}$$

In equation (11) the Ψ^k subscripts also mean partial differentiation with respect to the Ψ^k.

3. Maximal symmetry group of the Dirac equation

We now solve equation (11) for the Dirac equation of finite mass spin-$\frac{1}{2}$ particles and obtain the corresponding maximal symmetry group. The four-component Dirac Ψ satisfies

$$\partial\Psi/\partial\tau + \boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\Psi + (\mathrm{i}mc/\hbar)\beta\Psi = 0 \tag{12}$$

where

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} \qquad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \qquad \text{and} \qquad \tau=ct \tag{13}$$

$\boldsymbol{\sigma}$ being the Pauli matrices and I the 2×2 identity matrix. We thus have four equations $\Delta^m=0$, $m=1, 2, 3, 4$ corresponding to equation (9) with

$$\begin{aligned}&a^0_{11}=a^0_{22}=-a^0_{33}=-a^0_{44}=-\mathrm{i}mc/\hbar \qquad a^2_{14}=a^2_{32}=-a^2_{23}=-a^2_{41}=\mathrm{i}\\ &a^1_{14}=a^1_{23}=a^1_{32}=a^1_{41}=a^3_{13}=a^3_{31}=-a^3_{24}=-a^3_{42}=-1\end{aligned} \tag{14}$$

with all the other coefficients being zero.

Equating to zero the different coefficients of $\Psi^m_\alpha\Psi^k_\beta$, we obtain

$$\xi^\alpha_{\Psi^m}=\xi^\tau_{\Psi^m}=0 \qquad \alpha=1,2,3 \qquad \text{and} \qquad m=1,2,3,4. \tag{15}$$

Equating to zero the different coefficients of Ψ^m_α, we obtain

$$\begin{aligned}&\xi^\alpha_\beta+\xi^\beta_\alpha=0 \qquad \alpha\neq\beta \qquad \alpha,\beta=1,2,3\\ &\xi^\tau_\alpha-\xi^\alpha_\tau=\xi^\tau_\tau-\xi^\alpha_\alpha=0 \qquad \alpha=1,2,3\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\varphi_{1;\Psi^2}=\varphi_{3;\Psi^4}=-\tfrac{1}{2}(\xi^1_3-\mathrm{i}\xi^2_3)\\ &\varphi_{1;\Psi^4}=\varphi_{3;\Psi^2}=\tfrac{1}{2}(\xi^1_\tau-\mathrm{i}\xi^2_\tau)\\ &\varphi_{2;\Psi^1}=\varphi_{4;\Psi^3}=\tfrac{1}{2}(\xi^1_3+\mathrm{i}\xi^2_3)\\ &\varphi_{2;\Psi^3}=\varphi_{4;\Psi^1}=\tfrac{1}{2}(\xi^1_\tau+\mathrm{i}\xi^2_\tau)\\ &\varphi_{1;\Psi^1}=\varphi_{3;\Psi^3}=\varphi_{4;\Psi^4}+\mathrm{i}\xi^1_2=\varphi_{2;\Psi^2}+\mathrm{i}\xi^1_2\\ &\varphi_{1;\Psi^3}=\varphi_{3;\Psi^1}=\varphi_{4;\Psi^2}+\xi^\tau_3=\varphi_{2;\Psi^4}+\xi^\tau_3.\end{aligned} \tag{17}$$

Equating to zero the terms independent of Ψ^m_α, we obtain

$$\begin{aligned}&\varphi_{1;\tau}+\varphi_{3;3}+\varphi_{4;1}-\mathrm{i}\varphi_{4;2}+(\mathrm{i}mc/\hbar)\varphi_1\\ &\quad+(\mathrm{i}mc/\hbar)[(\xi^\tau_\tau-\varphi_{1;\Psi^1})\Psi^1+\tfrac{1}{2}(\xi^1_3-\mathrm{i}\xi^2_3)\Psi^2\\ &\quad+(\varphi_{1;\Psi^3}-\xi^\tau_3)\Psi^3-\tfrac{1}{2}(\xi^\tau_1-\mathrm{i}\xi^\tau_2)\Psi^4]=0\\ &\varphi_{2;\tau}-\varphi_{4;3}+\varphi_{3;1}+\mathrm{i}\varphi_{3;2}+(\mathrm{i}mc/\hbar)\varphi_2\\ &\quad+(\mathrm{i}mc/\hbar)[-\tfrac{1}{2}(\xi^1_3+\mathrm{i}\xi^2_3)\Psi^1+(\xi^\tau_\tau-\varphi_{2;\Psi^2})\Psi^2\\ &\quad+(\varphi_{2;\Psi^3}-\xi^\tau_1-\mathrm{i}\xi^\tau_2)\Psi^3+(\varphi_{2;\Psi^4}+\xi^\tau_3)\Psi^4]=0\\ &\varphi_{3;\tau}+\varphi_{1;3}+\varphi_{2;1}-\mathrm{i}\varphi_{2;2}-(\mathrm{i}mc/\hbar)\varphi_3\\ &\quad+(\mathrm{i}mc/\hbar)[(\xi^\tau_3-\varphi_{3;\Psi^1})\Psi^1+(\xi^\tau_1-\mathrm{i}\xi^\tau_2-\varphi_{3;\Psi^2})\Psi^2\\ &\quad+(\varphi_{3;\Psi^3}-\xi^\tau_\tau)\Psi^3+\varphi_{3;\Psi^4}\Psi^4]=0\\ &\varphi_{4;\tau}-\varphi_{2;3}+\varphi_{1;1}+\mathrm{i}\varphi_{1;2}-(\mathrm{i}mc/\hbar)\varphi_4\\ &\quad+(\mathrm{i}mc/\hbar)[(\xi^\tau_1+\mathrm{i}\xi^\tau_2-\varphi_{4;\Psi^1})\Psi^1-(\xi^\tau_3+\varphi_{4;\Psi^2})\Psi^2\\ &\quad+\varphi_{4;\Psi^3}\Psi^3+(\varphi_{4;\Psi^4}-\xi^\tau_\tau)\Psi^4]=0.\end{aligned} \tag{18}$$

From equation (16) it follows that

$$\partial^3\xi^\alpha/(\partial x^1)^k(\partial x^2)^l(\partial x^3)^m(\partial\tau)^n=\partial^3\xi^\tau/(\partial x^1)^k(\partial x^2)^l(\partial x^3)^m(\partial\tau)^n=0 \tag{19}$$

$k, l, m, n =$ non-negative integers, $k+l+m+n=3$, $\alpha = 1, 2, 3$. Equations (19) and (15) give us

$$\xi^\alpha = a_\alpha + bx^\alpha - \sum_{\beta\gamma} e_{\alpha\beta\gamma} b_\beta x^\gamma + b_{3+\alpha}\tau - c_\alpha(r^2-\tau^2) + 2x^\alpha\left(\sum_\beta c_\beta x^\beta + c_4\tau\right)$$
$$\xi^\tau = a_4 + b\tau + \sum_\alpha b_{3+\alpha}x^\alpha + c_4(r^2-\tau^2) + 2\tau\left(\sum_\alpha c_\alpha x^\alpha + c_4\tau\right) \tag{20}$$

where $r^2 = \Sigma_\alpha (x^\alpha)^2$, $e_{\alpha\beta\gamma} =$ permutation symbol, and a, b and c are constants. Equations (17) and (15) give us

$$\varphi_{k;\Psi^m\Psi^n} = 0 \qquad \text{and thus} \qquad \varphi_k(x^\alpha, \tau, \Psi) = \varphi_k^0(x^\alpha, \tau) + \sum_m \varphi_{k;\Psi^m}(x^\alpha, \tau)\Psi^m. \tag{21}$$

Using equations (20) and (21) and equating to zero the coefficients of different Ψ^k in equation (18) we obtain

$$c_4 = c_\alpha = b = 0 \qquad \varphi_{1;\Psi^3} = \xi_3^\tau/2 = b_6/2 \qquad \varphi_{1;\Psi^1} = c = \text{constant}.$$

We thus obtain

$$\xi^\alpha = a_\alpha - \sum_{\beta\gamma} e_{\alpha\beta\gamma} b_\beta x^\gamma + b_{3+\alpha}\tau \qquad \xi^\tau = a_4 + \sum_\alpha b_{3+\alpha}x^\alpha$$
$$\varphi_1 = \varphi_1^0 + c\Psi^1 + (b_2 + \mathrm{i}b_1)\Psi^2/2 + b_6\Psi^3/2 + (b_4 - \mathrm{i}b_5)\Psi^4/2$$
$$\varphi_2 = \varphi_2^0 - (b_2 - \mathrm{i}b_1)\Psi^1/2 + (c - \mathrm{i}b_3)\Psi^2/2 + (b_4 + \mathrm{i}b_5)\Psi^3/2 - b_6\Psi^4/2 \tag{22}$$
$$\varphi_3 = \varphi_3^0 + b_6\Psi^1/2 + (b_4 - \mathrm{i}b_5)\Psi^2/2 + c\Psi^3 + (b_2 + \mathrm{i}b_1)\Psi^4/2$$
$$\varphi_4 = \varphi_4^0 + (b_4 + \mathrm{i}b_5)\Psi^1/2 - b_6\Psi^2/2 - (b_2 - \mathrm{i}b_1)\Psi^3/2 + (c - \mathrm{i}b_3)\Psi^4.$$

Equating to zero the terms independent of Ψ^k in equation (18), we obtain

$$\varphi_{l;\tau}^0 + \sum_{n\lambda} \alpha_{ln}^\lambda \varphi_{n;\lambda}^0 + (\mathrm{i}mc/\hbar)\sum_n \beta_{ln}\varphi_n^0 = 0. \tag{23}$$

It is evident from equation (23) that φ_k^0 do not contain the a, b and c of equation (22). Thus the generators originating from φ_k^0 are independent of those arising from ξ^α, ξ^τ and $\varphi_{k;\Psi^m}$. We therefore concentrate our attention first on these latter generators. The eleven generators given by equation (22) are

$$X^\lambda = -\mathrm{i}\partial/\partial x^\lambda \qquad X^\tau = -\mathrm{i}\partial/\partial\tau$$
$$X_{\mathrm{R}}^\lambda = -\mathrm{i}\sum_{\mu\nu} e_{\lambda\mu\nu}x^\mu\partial/\partial x^\nu - \tfrac{1}{2}\sum_{km}\tilde{\alpha}_{km}^{\lambda *}\Psi^k\partial/\partial\Psi^m$$
$$X_{\mathrm{L}}^\lambda = (\tau\partial/\partial x^\lambda + x^\lambda\partial/\partial\tau) + \tfrac{1}{2}\sum_{km}\alpha_{km}^{\lambda *}\Psi^k\partial/\partial\Psi^m \tag{24}$$
$$X^\Psi = \sum_m \Psi^m\partial/\partial\Psi^m$$

where

$$\tilde{\boldsymbol{\alpha}} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \qquad \lambda, \mu, \nu = 1, 2, 3 \qquad \text{and} \qquad k, m = 1, 2, 3, 4.$$

The non-vanishing commutation relations are

$$[X^\lambda, X_{\mathrm{R}}^\mu] = \mathrm{i}\sum_\nu e_{\lambda\mu\nu}X^\nu \qquad [X^\lambda, X_{\mathrm{L}}^\mu] = \delta_{\lambda\mu}X^\tau$$
$$[X^\tau, X_{\mathrm{L}}^\lambda] = X^\lambda \qquad [X_{\mathrm{R}}^\lambda, X_{\mathrm{R}}^\mu] = \mathrm{i}\sum_\nu e_{\lambda\mu\nu}X_{\mathrm{R}}^\nu \tag{25}$$
$$[X_{\mathrm{R}}^\lambda, X_{\mathrm{L}}^\mu] = \mathrm{i}\sum_\nu e_{\lambda\mu\nu}X_{\mathrm{L}}^\nu \qquad [X_{\mathrm{L}}^\lambda, X_{\mathrm{L}}^\mu] = \mathrm{i}\sum_\nu e_{\lambda\mu\nu}X_{\mathrm{R}}^\nu.$$

Physically X^λ and X^τ denote translations along x^λ and τ coordinates; X_R^λ denotes rotation about x^λ axis together with transformation of Ψ^k; X_L^λ is the Lorentz boost along x^λ coupled with transformation of Ψ^k; and X^Ψ denotes scale transformation of Ψ^k. These eleven operators form a subgroup $\bar{G}$ of the maximal symmetry group G of the Dirac equation. The proper subgroup H of $\bar{G}$ consisting of all the generators except X^Ψ is isomorphic to the Poincaré group.

We now turn to equation (23). Since we are investigating Lie group structures, we consider analytic solutions

$$\varphi_m^0 = \sum_{\{n_i\}} (n!/n_1!n_2!n_3!n_4!)K_m(n_1, n_2, n_3, n_4)(x^1)^{n_1}(x^2)^{n_2}(x^3)^{n_3}(\tau)^{n_4} \tag{26}$$

with $n = n_1 + n_2 + n_3 + n_4$, where $K_m(n_1, n_2, n_3, n_4)$ are constants. Equation (23) gives us the following recursion relation among these constants:

$$K_l(n_1, n_2, n_3, n_4+2) = K_l(n_1+2, n_2, n_3, n_4) + K_l(n_1, n_2+2, n_3, n_4) + K_l(n_1, n_2, n_3+2, n_4) - [n!/(n+2)!](mc/\hbar)^2 K_l(n_1, n_2, n_3, n_4)$$

with $n = n_1 + n_2 + n_3 + n_4$ and

$$K_l(n_1, n_2, n_3, 1) = -\sum_{p\mu} \alpha_{pl}^{\mu *} K_p(n_\mu + 1, \{n_{\mu'}\}, 0) - [1/(n+1)](imc/\hbar) \sum_p \beta_{pl}^* K_p(n_1, n_2, n_3, 0) \tag{27}$$

with $n = n_1 + n_2 + n_3$.

Out of the total number $2[(n+3)!/n!]/3$ of constants $K_m(n_1, n_2, n_3, n_4)$, homogeneous of degree $n = n_1 + n_2 + n_3 + n_4$, the number of independent constants is thus equal to $2(n+1)(n+2)$. These mutually commuting independent generators are infinite in number, forming the infinite parameter Lie group G_∞. G_∞ is an invariant subgroup of G such that $G/G_\infty \approx \bar{G}$. Thus G is the semi-direct product $G = G_\infty \circledS \bar{G}$.

We now obtain the general form of the generators of G_∞. Using equation (27), after some combinatorial calculation, we obtain

$$\sum_n \varphi_n^0 X_n = \sum_{n_1}\sum_{n_2}\sum_{n_3} [(n_1+n_2+n_3)!/n_1!n_2!n_3!] \sum_n K_n(n_1, n_2, n_3, 0) X_n(n_1, n_2, n_3)$$

where

$$\begin{aligned} X_l(n_1, n_2, n_3) = \sum_{s=0}^{[n/2]} \sum_{s_1+s_2+s_3=s} & (n_1!n_2!n_3!s!/s_1!s_2!s_3!) \\ \times \Bigg(& X_l f_s^1(\tau)[\tau^{2s}/(2s)!] \prod_\mu [(x^\mu)^{n_\mu - 2s_\mu}/(n_\mu - 2s_\mu)!] \\ & - (imc/\hbar) f_s^2(\tau)[\tau^{2s+1}/(2s+1)!] \sum_p \beta_{lp}^* X_p \prod_\mu [(x^\mu)^{n_\mu - 2s_\mu}/(n_\mu - 2s_\mu)!] \\ & - f_s^2(\tau)[\tau^{2s+1}/(2s+1)!] \sum_{p\lambda} \alpha_{lp}^{\lambda *} X_p [(x^\lambda)^{n_\lambda - 2s_\lambda - 1}/(n_\lambda - 2s_\lambda - 1)!] \\ & \times \prod_{\mu \neq \lambda} [(x^\mu)^{n_\mu - 2s_\mu}/(n_\mu - 2s_\mu)!] \Bigg) \end{aligned}$$

with

$$f_s^1(\tau) = \sum_{r=0}^{\infty} [(2s)!(r+s)!/(2s+2r)!s!r!](imc\tau/\hbar)^{2r}$$

and

$$f_s^2(\tau)=[(2s+1)/(mc\tau/\hbar)^{2s+1}]\int_0^\tau (mc\tau/\hbar)^{2s}f_s^1(\tau)\,\mathrm{d}(mc\tau/\hbar). \tag{28}$$

Here we have used $[x]$ as the integral part of any positive real number x and have written X_l for $-\mathrm{i}\partial/\partial\Psi^l$.

The first few generators of G_∞ are

$$\begin{aligned}
X_l^0&=X_l\cos(mc\tau/\hbar)-\mathrm{i}\sum_n \beta_{ln}^* X_n\sin(mc\tau/\hbar)\\
X_l^\mu&=x^\mu X_l^0-(\hbar/mc)\sum_n \alpha_{ln}^{\mu *}X_n\sin(mc\tau/\hbar)\\
X_l^{\lambda\mu}&=x^\lambda x^\mu X_l^0-(\hbar/mc)\sin(mc\tau/\hbar)\left(x^\lambda\sum_n \alpha_{ln}^{\mu *}X_n+x^\mu\sum_n \alpha_{ln}^{\lambda *}X_n\right)\\
&\quad+\delta_{\lambda\mu}(\hbar\tau/mc)\left(X_l\sin(mc\tau/\hbar)+\mathrm{i}\sum_n \beta_{ln}^* X_n\cos(mc\tau/\hbar)\right.\\
&\quad\left.-\mathrm{i}(\hbar/mc\tau)\sum_n \beta_{ln}^* X_n\sin(mc\tau/\hbar)\right)
\end{aligned} \tag{29a}$$

with non-vanishing commutation relations

$$\begin{aligned}
&[X_l^0,X^\tau]=(mc/\hbar)\sum_n \beta_{ln}^* X_n^0 \qquad [X_l^0,X^\Psi]=X_l^0\\
&[X_l^0,X_\mathrm{R}^\mu]=-\tfrac12\sum_n \tilde\alpha_{ln}^{\mu *}X_n^0 \qquad [X_l^0,X_\mathrm{L}^\mu]=(\mathrm{i}mc/\hbar)\sum_n \beta_{ln}^* X_n^\mu+\tfrac12\sum_n \alpha_{ln}^{\mu *}X_n^0\\
&[X_l^\lambda,X^\mu]=\mathrm{i}\delta_{\lambda\mu}X_l^0 \qquad [X_l^\mu,X^\tau]=(mc/\hbar)\sum_n \beta_{ln}^* X_n^\mu-\mathrm{i}\sum_n \alpha_{ln}^{\mu *}X_n^0\\
&[X_l^\mu,X^\Psi]=X_l^\mu \qquad [X_l^\lambda,X_\mathrm{R}^\mu]=\mathrm{i}\sum_\nu e_{\lambda\mu\nu}X_l^\nu-\tfrac12\sum_n \tilde\alpha_{ln}^{\mu *}X_n^\lambda\\
&[X_l^\lambda,X_\mathrm{L}^\mu]=(\mathrm{i}mc/\hbar)\sum_n \beta_{ln}^* X_n^{\lambda\mu}+\sum_n \alpha_{ln}^{\lambda *}X_n^\mu+\tfrac12\sum_n \alpha_{ln}^{\mu *}X_n^\lambda .
\end{aligned} \tag{29b}$$

4. Maximal symmetry group of the Weyl equation

In this section we shall obtain the maximal symmetry group of the Weyl equation for zero mass spin-$\frac12$ particles. The two-component Weyl equation satisfies

$$\partial\Psi/\partial\tau+\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\Psi=0 \qquad \tau=ct \tag{30a}$$

where $\boldsymbol{\sigma}$ are Pauli matrices. We thus have two equations $\Delta^m=0$, $m=1,2$, corresponding to equation (9), with

$$a_{mk}^0=0 \qquad a_{11}^3=a_{12}^1=a_{21}^1=-a_{22}^3=-1 \qquad a_{12}^2=-a_{21}^2=\mathrm{i} \tag{30b}$$

the other coefficients being zero.

Equating to zero the different coefficients of $\Psi_\alpha^m\Psi_\beta^k$, we obtain

$$\xi_{\Psi^m}^\tau=\xi_{\Psi^m}^\alpha=0 \qquad \alpha=1,\,2,\,3 \qquad m=1,\,2. \tag{31}$$

Equating to zero the different coefficients of Ψ_α^m, we obtain

$$\xi_\beta^\alpha+\xi_\alpha^\beta=0 \qquad \alpha\neq\beta \qquad \xi_\alpha^\tau-\xi_\tau^\alpha=\xi_\tau^\tau-\xi_\alpha^\alpha=0 \qquad \alpha,\beta=1,2,3$$

and

$$\varphi_{1;\Psi^2}=\tfrac{1}{2}(\xi_1^\tau-\mathrm{i}\xi_2^\tau)+\tfrac{1}{2}(\xi_1^3-\mathrm{i}\xi_2^3)$$
$$\varphi_{2;\Psi^1}=\tfrac{1}{2}(\xi_1^\tau+\mathrm{i}\xi_2^\tau)-\tfrac{1}{2}(\xi_1^3+\mathrm{i}\xi_2^3) \qquad (32)$$
$$\varphi_{1;\Psi^1}-\varphi_{2;\Psi^2}=\xi_3^\tau+\mathrm{i}\xi_2^1.$$

Equating to zero the terms independent of Ψ_α^m, we obtain

$$\varphi_{1;\tau}+\varphi_{1;3}+\varphi_{2;1}-\mathrm{i}\varphi_{2;2}=\varphi_{2;\tau}-\varphi_{2;3}+\varphi_{1;1}+\mathrm{i}\varphi_{1;2}=0. \qquad (33)$$

From equation (31) it follows that

$$\partial^3\xi^\alpha/(\partial x^1)^k(\partial x^2)^l(\partial x^3)^m(\partial\tau)^n=\partial^3\xi^\tau/(\partial x^1)^k(\partial x^2)^l(\partial x^3)^m(\partial\tau)^n=0 \qquad (34)$$

where $k+l+m+n=3$.

Equations (30) and (34) give us

$$\xi^\alpha=a_\alpha+bx^\alpha-\sum_{\beta\gamma}e_{\alpha\beta\gamma}b_\beta x^\gamma+b_{3+\alpha}\tau-c_\alpha(r^2-\tau^2)+2x^\alpha\left(\sum_\beta c_\beta x^\beta+c_4\tau\right)$$
$$\xi^\tau=a_4+b\tau+\sum_\alpha b_{3+\alpha}x^\alpha+c_4(r^2-\tau^2)+2\tau\left(\sum_\alpha c_\alpha x^\alpha+c_4\tau\right) \qquad (35)$$

where $r^2=\Sigma_\alpha\,(x^\alpha)^2$, $\alpha=1, 2, 3$ and a, b and c are constants. Equations (30) and (32) give us

$$\varphi_{m;\Psi^n\Psi^k}=0 \qquad \text{and thus} \qquad \varphi_m=\varphi_m^0(x^\alpha,\tau)+\sum_n\varphi_{m;\Psi^n}(x^\alpha,\tau)\Psi^n. \qquad (36)$$

Using equations (35) and (36) and equating to zero the coefficients of different Ψ^m in equation (33) we obtain

$$\varphi_{m;\Psi^n}=\left(C+\mathrm{i}b_3/2+b_6/2-3\sum_\mu c_\mu x^\mu-3c_4\tau\right)\delta_{mn}$$
$$+\sum_\mu\sigma_{mn}^\mu b_{3+\mu}/2+\mathrm{i}\sum_\mu\sigma_{mn}^\mu b_\mu/2+c_4\sum_\mu\sigma_{mn}^\mu x^\mu+\tau\sum_\mu\sigma_{mn}^\mu c_\mu$$
$$-\mathrm{i}\sum_{\lambda\mu\nu}e_{\lambda\mu\nu}\sigma_{mn}^\lambda c_\mu x^\nu \qquad (37)$$

where C is a constant. Equating to zero the terms independent of Ψ^m in equation (33), we obtain

$$\varphi_{1;\tau}^0+\varphi_{1;3}^0+\varphi_{2;1}^0-\mathrm{i}\varphi_{2;2}^0=\varphi_{2;\tau}^0-\varphi_{2;3}^0+\varphi_{1;1}^0+\mathrm{i}\varphi_{1;2}^0=0. \qquad (38)$$

From equation (38) it is again evident that φ_m^0 do not contain the constants a, b and c and thus the generators obtained from the solutions of equation (38) are independent of those arising from ξ^α, ξ^τ and $\varphi_{m;\Psi^n}$. In the latter category we obtain 16 generators corresponding to the proper subgroup $\bar{\mathrm{G}}$ of the maximal symmetry group G,

$$X^\alpha=-\mathrm{i}\partial/\partial x^\alpha \qquad X^\tau=-\mathrm{i}\partial/\partial\tau \qquad X^\Psi=\sum_m\Psi^m\partial/\partial\Psi^m$$
$$X_0=\tau\partial/\partial\tau+\sum_\alpha x^\alpha\partial/\partial x^\alpha-\tfrac{3}{2}X^\Psi$$
$$X_{\mathrm{R}}^\alpha=-\mathrm{i}\sum_{\beta\gamma}e_{\alpha\beta\gamma}x^\beta\partial/\partial x^\gamma-\tfrac{1}{2}\sum_{km}\sigma_{km}^{\alpha *}\Psi^k\partial/\partial\Psi^m \qquad (39)$$
$$X_{\mathrm{L}}^\alpha=\tau\partial/\partial x^\alpha+x^\alpha\partial/\partial\tau+\tfrac{1}{2}\sum_{km}\sigma_{km}^{\alpha *}\Psi^k\partial/\partial\Psi^m$$

$$X^\alpha_A = (i/2)(r^2 - \tau^2)X^\alpha + \tau X^\alpha_L + i\sum_{\beta\gamma} e_{\alpha\beta\gamma} x^\beta X^\gamma_R - \tfrac{3}{2}x^\alpha X^\Psi$$

$$X^\tau_A = -(i/2)(r^2 - \tau^2)X^\tau + \sum_\alpha x^\alpha X^\alpha_L - \tfrac{3}{2}\tau X^\Psi \qquad \alpha, \beta, \gamma = 1, 2, 3$$

with the non-vanishing commutation relations

$$\begin{aligned}
&[X^\alpha, X_0] = X^\alpha \qquad [X^\alpha, X^\beta_R] = i\sum_\gamma e_{\alpha\beta\gamma}X^\gamma \qquad [X^\alpha, X^\beta_L] = \delta_{\alpha\beta}X^\tau\\
&[X^\alpha, X^\beta_A] = -i\delta_{\alpha\beta}X_0 - \sum_\gamma e_{\alpha\beta\gamma}X^\gamma_R \qquad [X^\alpha, X^\tau_A] = -iX^\alpha_L\\
&[X^\tau, X_0] = X^\tau \qquad [X^\tau, X^\alpha_L] = X^\alpha \qquad [X^\tau, X^\alpha_A] = -iX^\alpha_L \qquad [X^\tau, X^\tau_A] = -iX_0\\
&[X_0, X^\alpha_A] = X^\alpha_A \qquad [X_0, X^\tau_A] = X^\tau_A \qquad [X^\alpha_R, X^\beta_R] = i\sum_\gamma e_{\alpha\beta\gamma}X^\gamma_R\\
&[X^\alpha_R, X^\beta_L] = i\sum_\gamma e_{\alpha\beta\gamma}X^\gamma_L \qquad [X^\alpha_R, X^\beta_A] = i\sum_\gamma e_{\alpha\beta\gamma}X^\gamma_A\\
&[X^\alpha_L, X^\beta_L] = i\sum_\gamma e_{\alpha\beta\gamma}X^\gamma_R \qquad [X^\alpha_L, X^\beta_A] = \delta_{\alpha\beta}X^\tau_A \qquad [X^\alpha_L, X^\tau_A] = X^\alpha_A.
\end{aligned} \tag{40}$$

It should be noted that the subgroup consisting of the generators other than X^Ψ is isomorphic to the conformal group (Bateman 1910, Fulton *et al* 1962, Gross 1964). The generator X^Ψ corresponds to the scale transformation $\Psi^m \to s\Psi^m$. It is also clear that X^α_R and X^α_L are not pure rotations and Lorentz boosts in the coordinate space, but the Ψ^m are also to be simultaneously transformed. However $X^\alpha_S = X^\alpha_R + X^\alpha_L$ is a symmetry operation without change in the Ψ^m. Physically X^α_S is a screw transformation, being a simultaneous rotation about the x^α axis and a Lorentz boost along the same direction.

We now determine the other generators of G given by equation (38). Because of our interest in the Lie group symmetry, we again use an expansion equation (26) and get the recursion relation

$$\begin{aligned}
K_m(n_1, n_2, n_3, n_4+2) &= K_m(n_1+2, n_2, n_3, n_4) + K_m(n_1, n_2+2, n_3, n_4)\\
&\quad + K_m(n_1, n_2, n_3+2, n_4)\\
K_m(n_1, n_2, n_3, 1) &= -\sum_{n\mu} \sigma^{\mu *}_{nm} K_n(n_\mu + 1, \{n_{\mu'}\}, 0) \qquad m = 1, 2.
\end{aligned} \tag{41}$$

Out of the total number of $\frac{1}{3}(n+3)!/n!$ constants $K_m(n_1, n_2, n_3, n_4)$, homogeneous of degree $n = n_1 + n_2 + n_3 + n_4$, the number of independent constants is thus equal to $(n+1)(n+2)$. Combinatorial calculation gives the independent generators

$$\begin{aligned}
X_m(n_1, n_2, n_3) = \sum_{s=0}^{[n/2]} \sum_{s_1+s_2+s_3=s} &(n_1!\,n_2!\,n_3!\,s!/s_1!\,s_2!\,s_3!)\\
&\times\Big(X_m[\tau^{2s}/(2s)!]\prod_\mu [(x^\mu)^{n_\mu - 2s_\mu}/(n_\mu - 2s_\mu)!]\\
&-\sum_{\mu p} \sigma^{\mu *}_{mp} X_p[\tau^{2s+1}/(2s+1)!][(x^\mu)^{n_\mu - 2s_\mu - 1}/(n_\mu - 2s_\mu - 1)!]\\
&\times \prod_{\nu \neq \mu} [(x^\nu)^{n_\nu - 2s_\nu}/(n_\nu - 2s_\nu)!]\Big)
\end{aligned} \tag{42}$$

where $n = n_1 + n_2 + n_3$.

These mutually commuting independent generators again form an infinite parameter Lie group G_∞, which is an invariant subgroup of G such that $G/G_\infty \approx \bar{G}$. Thus G is again a semi-direct product $G = G_\infty \circledS \bar{G}$. The first few generators of G_∞ are

$$X_m^0 = X_m \qquad X_m^\mu = x^\mu X_m^0 - \tau \sum_n \sigma_{mn}^{\mu *} X_n^0$$
$$X_m^{\lambda\mu} = x^\lambda x^\mu X_m^0 - x^\mu \tau \sum_n \sigma_{mn}^{\lambda *} X_n^0 - x^\lambda \tau \sum_n \sigma_{mn}^{\mu *} X_n^0 + \tau^2 \delta_{\lambda\mu} X_m^0, \ldots \tag{43}$$

with the non-vanishing commutation relations

$$[X_m^0, X^\Psi] = X_m^0 \qquad [X_m^0, X_0] = -\tfrac{3}{2} X_m^0 \qquad [X_m^0, X_R^\mu] = -\tfrac{1}{2} \sum_n \sigma_{mn}^{\mu *} X_n^0$$
$$[X_m^0, X_L^\mu] = \tfrac{1}{2} \sum_n \sigma_{mn}^{\mu *} X_n^0 \qquad [X_m^0, X_A^\lambda] = -\tfrac{3}{2} X_m^\lambda - (\mathrm{i}/2) \sum_{\mu\nu n} e_{\lambda\mu\nu} \sigma_{mn}^{\nu *} X_n^\mu$$
$$[X_m^0, X_A^\tau] = \tfrac{1}{2} \sum_{\mu n} \sigma_{mn}^{\mu *} X_n^\mu \qquad [X_m^\lambda, X^\mu] = \mathrm{i} \delta_{\lambda\mu} X_m^0$$
$$[X_m^\mu, X^\tau] = -\mathrm{i} \sum_n \sigma_{mn}^{\mu *} X_n^0 \qquad [X_m^\mu, X^\Psi] = X_m^\mu$$
$$[X_m^\mu, X_0] = -\tfrac{5}{2} X_m^\mu \qquad [X_m^\lambda, X_R^\mu] = \mathrm{i} \sum_\nu e_{\lambda\mu\nu} X_m^\nu - \tfrac{1}{2} \sum_n \sigma_{mn}^{\mu *} X_n^\lambda \tag{44}$$
$$[X_m^\lambda, X_L^\mu] = \tfrac{1}{2} \sum_n \sigma_{mn}^{\mu *} X_n^\lambda + \sum_n \sigma_{mn}^{\lambda *} X_n^\mu$$
$$[X_m^\lambda, X_A^\mu] = \tfrac{1}{2} \delta_{\lambda\mu} \sum_\nu X_m^{\nu\nu} - \tfrac{5}{2} X_m^{\lambda\mu} - (\mathrm{i}/2) \sum_{\nu\gamma} e_{\mu\nu\gamma} \sigma_{mn}^{\gamma *} X_n^{\nu\lambda}$$
$$[X_m^\lambda, X_A^\tau] = \tfrac{1}{2} \sum_{\mu n} \sigma_{mn}^{\lambda *} X_n^{\mu\mu} + \tfrac{1}{2} \sum_{\mu n} \sigma_{mn}^{\mu *} X_n^{\mu\lambda}, \ldots .$$

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